Competify Hub

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MATH LEVEL 3

GRADES 9-12

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Q1: Let O be the center of an ellipse. Let A, B, and P be points on the ellipse such that OA is a semimajor axis, OB is a semiminor axis, and ∠POA = 45°. If the area of the sector of the ellipse bounded by rays OA and OP is 1/6 of the area of the ellipse, find OA/OB in simplest radical form.

Q2: In ΔABC with AB = 13, BC = 14, and CA = 15, there is an ellipse inscribed in ΔABC such that one focus is the orthocenter of ΔABC. Find the length of the major axis of this ellipse as a common fraction.

S1:

$(\sqrt{6})/2$. Let f be the transformation that stretches the plane by a factor of OB/OA in the direction of OA, and let Q be the projection of P onto OA.
Also, let A' = f(A), P' = f(P), and Q' = f(Q).
Note that under f, the ellipse becomes a circle with center O and radius OB, so $10\angle P'OA' = (360°)(1/6) = 60°$ because of the area condition.
Therefore, OA/OB = OA/OA' = OQ/OQ' = (cos 60°)/(cos 45°) = $(\sqrt{3})/(\sqrt{2}) = (\sqrt{6})/2$.

S2:

65/8. Let H and O be the orthocenter and circumcenter of ΔABC, respectively.

Since H is one of the foci, O must be the other focus because H and O are isogonal conjugates.

Now, let H' be the reflection of H over BC. It is well-known that H' lies on the circumcircle of ΔABC, so the length of the major axis is OH' = (13)(14)(15)/(4[ABC]).

The semiperimeter of ΔABC is (13 + 14 + 15)/2 = 42/2 = 21, so by Heron's Formula, we get [ABC] = $\sqrt{(21 * (21 - 13) * (21 - 14) * (21 - 15))} = \sqrt{(21 * 8 * 7 * 6)} = 84$.

Thus, the length of the major axis is (13)(14)(15)/(4 * 84) = 65/8.

Q3

If A is a point on the graph of $y = x^2$ and B is a point on the graph of $y = 2x - 5$, find the minimum possible distance from A to B. Express your answer as a common fraction in simplest radical form.

Q4:

If x, y, and z are randomly and uniformly picked from the interval [0, 1], what is the probability that $1 \le x + y + z \le 2$? Express your answer as a common fraction.

S3

4(√5)/5. Let A = (a_x, a_y) and let B = (b_x, b_y). Also, let C = (a_x, 2(a_x) - 5) be the point on the graph of y = 2x - 5 that has the same x coordinate as A, and let D = (a_x, b_y).

Note that the distance is minimized when line AB is perpendicular to the line y = 2x - 5.

Then, ΔABC ~ ΔBDC, and since the slope of BC is 2, we get BC = √(BD^2 + (2BD)^2) = √(5BD^2) = BD(√5). Thus, AB = AC(√5).

Now, note that AC = (a_x)^2 - (2(a_x) - 5) = (a_x)^2 - 2(a_x) + 5 = (a_x - 1)^2 + 4, so the minimum of AC is 4.

Thus, the answer is 4(√5)/5.

S4

2/3. If we graph the problem in 3D space, we will see that the desirable volume is just a 1x1x1 cube with two triangular pyramids removed.

The volume of each of these pyramids is (((1 * 1)/2) * 1)/3 = 1/6.

Thus, the answer is (1 - 2(1/6))/1 = 2/3.

Q5

What are the last three digits of 256^{625}?

Q6:

ΔABC has a right angle at A, and all of its side lengths are integers. If $AB = 30$, what is the sum of all possible values of BC?

S5

376. We can consider 256^625 mod 8 and 256^625 mod 125. Obviously, 256^625 is 0 mod 8. However, 256^625 = 2^5000 = 1 mod 125 since phi(125) = 100. Hence, the answer is 376 by the Chinese Remainder Theorem.

S6

388. Let a = BC and b = CA.

Then, a^2 - b^2 = 30^2, or (a + b)(a - b) = 900. Now, let c = a + b and let d = a - b.

Since c and d must be the same parity with c > d to produce a valid pair for a and b, these are the only valid values:

c	*d*	*a*	*b*
450	*2*	*226*	*224*
150	*6*	*78*	*72*
90	*10*	*50*	*40*
50	*18*	*34*	*16*

Thus, the answer is 226 + 78 + 50 + 34 = 388.

Q7

You are playing a coin game. If you flip heads, you earn a dollar and keep playing the game. If you flip tails, you must stop the game. How many dollars are you expected to earn?

Q8

For every 9999 good cars produced at a factory, 1 car is defective. To detect defective cars, a defective car detector scans all the cars and it returns the correct result 90% of the time. If the defective car detector detects a car as defective, what is the probability that the defective car detector detected a defective car? Express your answer as a common fraction.

S7

2. Let v be the expected value of playing the game.
Then, v = (1/2)(v + 1), because there is a 1/2 chance that you earn one dollar and keep playing the game and also a 1/2 chance you stop.
Solving this equation, we get v = 2

S8

Initially, the ratio of good cars to defective cars is 9999:1.

However, since defective cars get detected as defective (90%)/(10%) = 9 times more often than good cars, the ratio of good cars detected as defective to defective cars detected as defective is 1111:1.

Therefore, there is a 1/(1 + 1111) = 1/1112 chance that a car detected as defective is actually defective.

Q9

A hyperbola with foci at A = (-6, 7) and B = (6, -2) is tangent to the x axis. Find the minimum of PA - PB, where P is a point on the hyperbola.

Q10

If ABCD is a quadrilateral such that AC = 2, $\angle B = 60°$, and $\angle D = 45°$, what is the maximum possible area of ABCD? Express your answer in simplest radical form.

S9

-13. Let Q be the point on the x axis such that |QA - QB| is maximized. Then, Q is the tangency point of the hyperbola. However, if we let B' = (6, 2) be B reflected over the x axis, then |QA - QB| = |QA - QB'|, and by the Triangle Inequality, |QA - QB'| is maximized when A, B', and Q are collinear. Therefore, since |QA - QB'| = AB' = 13, we must have |PA - PB| = for all P on the hyperbola. Selecting P on the branch closer to A gives the minimum of -13.

S10

$1 + \sqrt{2} + \sqrt{3}$. The locus of all points B such that $\angle ABC = 60°$ is a pair of arcs of circles that contain A and C.

Since the area of a triangle grows with its height, [ABC] is maximized when B is at either arc midpoint of A and C. This results in ΔABC being equilateral, so the maximum of [ABC] is $\sqrt{3}$.

The locus of all points D such that $\angle ADC = 45°$ is a different pair of arcs of circles that contain A and C.

Since the area of a triangle grows with its height, [ADC] is maximized when D is at either arc midpoint of A and C. If we let O be the circumcenter of ΔADC, we notice that ΔAOC is an isosceles right triangle, so $AO = CO = DO = \sqrt{2}$, and the height from O to AC is 1.

Therefore, the maximum of $[ADC] = 1 + \sqrt{2}$.

Putting ΔABC and ΔADC together such that B and D lie on opposite sides of AC, we get that the maximum of [ABCD] is $1 + \sqrt{2} + \sqrt{3}$.