COMPETIFY HUB

FUELING AMBITION, FORGING PATHS

MATH LEVEL 3

GRADES 9-12

PROVIDING FREE RESOURCES FOR ALL

Demo Set 2

Q11

Let the roots of $x^4 - 3x^3 + 7x^2 + 8x - 1$ be r, s, t, and u. Find the sum of the coefficients of a monic quartic polynomial that has roots r^3, s^3, t^3, and u^3.

Q12

Let ΔABC be an equilateral triangle with points D, E, and F, on sides AB, BC, and CA, respectively. If P is a point in the interior of ΔABC such that $PD = 5$, $PE = 6$, $PF = 7$, and $\angle PDA = \angle PEB = \angle PFC$, find [DEF]. Express your answer as a common fraction in simplest radical form.

S11

1764. Let $P(x) = x^4 - 3x^3 + 7x^2 + 8x - 1$, and let $w = e^{(2\pi i/3)}$. Note that we are simply being asked to find
$(1 - r^3)(1 - s^3)(1 - t^3)(1 - u^3)$,
which can be factored into
$(1 - r)(1 - s)(1 - t)(1 - u)(w - r)(w - s)(w - t)(w - u)(w^2 - r)(w^2 - s)(w^2 - t)(w^2 - u) = P(1)P(w)P(w^2)$
$= (1^4 - 3(1^3) + 7(1^2) + 8(1) - 1)(w^4 - 3w^3 + 7w^2 + 8w - 1)(w^8 - 3w^6 + 7w^4 + 8w^2 - 1)$
$= 12(7w^2 + 9w - 4)(9w^2 + 7w - 4) = 12(2w - 11)(2w^2 - 11) = 12(4w^3 - 22w^2 - 22w + 121)$
$= 12(-22w^2 - 22w + 125) = 12(147) = 1764$.

S12

$(107\sqrt{3})/4$. Since $\angle PDA = \angle PEB = \angle PFC$, we get that quadrilaterals ADPF, BEPD, and CFPE are cyclic, so $\angle DPE = \angle EPF = \angle FPD = 120°$.

Thus,

$[DEF] = [DPE] + [EPF] + [FPD] = (5 * 6 * \sin 120°)/2 + (6 * 7 * \sin 120°)/2 + (7 * 5 * \sin 120°)/2$

$= (5 * 6 + 6 * 7 + 7 * 5)(\sin 120°)/2 = (30 + 42 + 35)(\sqrt{3})/4 = (107\sqrt{3})/4$.

Q13

Let ΔABC be a triangle with side lengths AB = 13, BC = 15, and CA = 14. Equilateral triangles ABX and BCY are constructed such that they do not overlap with ΔABC. If the circumcircles of ABX and BCY intersect at P ≠ B, find (PA + PB + PC)^2. Express your answer in simplest radical form.

Q14

Alice flips 5 coins, while Bob flips 7 coins. What is the probability that they flip the same number of heads? Express your answer as a common fraction.

S13

$295 + 168\sqrt{3}$. Let $a = PA$, $b = PB$, and $c = PC$. Note that $\angle APB = \angle BPC = \angle CPA = 120°$. Since $\cos 120° = -1/2$, by the Law of Cosines, we get

$a^2 + ab + b^2 = 13^2$

$b^2 + bc + c^2 = 15^2$

$c^2 + ca + a^2 = 14^2$.

Adding these equations gives us

$2a^2 + ab + 2b^2 + bc + 2c^2 + ca = 13^2 + 15^2 + 14^2$

$2a^2 + ab + 2b^2 + bc + 2c^2 + ca = 590.$ (1)

Now, note that by Heron's Formula, $[ABC] = \sqrt{21 * (21 - 13) * (21 - 14) * (21 - 15)} = \sqrt{21 * 8 * 7 * 6} = 84$. However, we also know that

$[ABC] = [APB] + [BPC] + [CPA] = (ab \sin 120°)/2 + (bc \sin 120°)/2 + (ca \sin 120°)/2$

$= (ab + bc + ca)(\sin 120°)/2 = (ab + bc + ca)(\sqrt{3})/4$.

Therefore,

$84 = (ab + bc + ca)(\sqrt{3})/4$

$112\sqrt{3} = ab + bc + ca$

$336\sqrt{3} = 3ab + 3bc + 3ca.$ (2)

Adding (1) and (2) gives us

$2a^2 + 4ab + 2b^2 + 4bc + 2c^2 + 4ca = 590 + 336\sqrt{3}$

$2(a + b + c)^2 = 590 + 336\sqrt{3}$.

$(a + b + c)^2 = 295 + 168\sqrt{3}$.

S14

Solution: 99/512. Note that Alice has the same probability of flipping x heads and 5 - x tails as flipping x tails and 5 - x heads. So, we can instead find the probability that Alice flips the same number of tails as Bob flips heads.

However, the number of ways this can happen is simply 12C5 since there are always exactly 5 heads distributed among Alice and Bob.

Thus, the answer is $(12C5)/(2^{12}) = 792/4096 = 99/512$.

Q15

Let f be a function that permutes the set {1, 2, ..., 6}. Also, define the order of f to be the least positive integer k such that f(f(...f(x)...)) = x when f is applied k times. If the order of f is 6, how many possible functions can f be?

Q16

Find all real solutions of x^4 - 5x^3 + 6x^2 - 5x + 1 = 0. Express your answer as a list, separated by commas, in simplest radical form.

S15

300. We can decompose f into cycles of length $a_1, a_2, \ldots, a_n$ with $a_1 + a_2 + \ldots + a_n = 6$ where n is the number of cycles in f.

Note that the order of f is just $\mathrm{lcm}(a_1, a_2, \ldots, a_n)$, which means one of the cycles must have length 3 or 6 and one of the cycles must have length 2 or 6. Therefore, the only possible configurations that work are a single cycle of length 6 or 3 cycles of lengths 3, 2, and 1. The first case has $5! = 120$ functions and the second case has $6 \cdot \binom{6}{2} \cdot 2 = 180$ functions, so our answer is 300.

S16

$2 + \sqrt{3}$, $2 - \sqrt{3}$. Since $x = 0$ is obviously not a solution, we can divide by x^2 to get

$$x^2 - 5x + 6 - 5/x + 1/(x^2) = 0.$$

Now, let $a = x + 1/x$, and rewriting the equation in terms of a gives us $a^2 - 5a + 4 = 0$, or $(a - 1)(a - 4) = 0$. Therefore, the solutions are when $a = 1$ and $a = 4$.

For $a = 1$, we get $1 = x + 1/x$, or $x^2 - x + 1 = 0$. However, this has complex solutions, so we do not consider it.

For $a = 4$, we get $4 = x + 1/x$ or $x^2 - 4x + 1 = 0$, which has roots $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

Q17

Find the sum of all integers n such that there are exactly 64 positive integers less than or equal to n that are relatively prime to n.

Q18

Let H, I, and O denote the orthocenter, circumcenter, and incenter, of ΔABC, respectively. If B, H, I, and C lie on a circle, and $\angle ABC = 50°$, find $\angle IBO$.

S17

1315. Since 64 is a power of 2 and phi(n) = 64, we can express n as (2^a)(p_1)(p_2)...(p_k), where a is a nonnegative integer and the p_x are distinct odd primes that are 1 more than a power of 2. We will proceed by casework on a.

When a = 0, the only possible n is 17 * 5 = 85.

When a = 1, the only possible n is 2 * 17 * 5 = 170.

When a = 2, the only possible n is 2^2 * 17 * 3 = 204.

When a = 3, the only possible n is 2^3 * 17 = 136.

When a = 4, the only possible n is 2^4 * 3 * 5 = 240.

When a = 5, the only possible n is 2^5 * 5 = 160.

When a = 6, the only possible n is 2^6 * 3 = 192.

When a = 7, the only possible n is 2^7 = 128.

When a >= 8, phi(n) >= 128, so this case is not possible.

Therefore, the answer is 85 + 170 + 204 + 136 + 240 + 160 + 192 + 128 = 1315.

S18

5°. Since ∠BHC = ∠BOC, we get 180° - ∠BAC = 2∠BAC, so ∠BAC = 60°. Therefore, I also lies on the circle with B, H, I, and C because ∠BIC = ∠BHC = ∠BOC = 120°.

Now, since ∠ABC = 50°, ∠IBC = 25°, and since ∠OBC = 90° - ∠BAC = 30°, ∠IBO = 5°.

Q19

Find the product of all values of k such that the ellipse defined by the equation x^2 + 4y^2 = 4 is tangent to the line x + y = k.

Q20

An alien has 20 fingers, all in a line. Before the alien ever says “orz”, all its fingers are raised. Every time the alien says “orz”, it puts down its leftmost raised finger, and it raises all the fingers to the left of this finger. How many fingers up does the alien have after saying “orz” 69420 times?

S19

-5. Suppose that the line and ellipse are tangent at a point P. Then, the sum of the coordinates of P is either minimized or maximized over all points on the ellipse.
Now, we can let P = (2 cos t, sin t) for some t. Note that
2 cos t + sin t = (√5)((2/(√5)) cos t + (1/(√5)) sin t) = (√5) cos(t - ø),
where ø is an angle such that cos ø = 2/(√5) and sin ø = 1/(√5). Therefore, the minimum and maximum are -√5 and √5, respectively, so the answer is -5.

S20

12. We can interpret the alien's fingers as a 20 digit binary number read right to left where a finger being raised is a 0 and a finger being down is a 1.

Then, the alien starts at 0 and each time it says "orz", the binary number increments by 1.

So, at the end, the binary number being shown is 69420, which is 00010000111100101100 as a 20 digit binary number. Since this number has 12 0s, the answer is 12.