Competify Hub

Fueling Ambition, Forging Paths

MATH LEVEL 3

GRADES 9-12

Providing Free Resources for All

Demo Set 3

Q21

A drunk man is in a field. Every 5 seconds, he takes a drunken step of 4 bottle lengths in a random direction. What is the expected value of the square of the number of bottle lengths from him to his starting point, after 10 minutes?

Q22

Quadrilateral ABCD has side lengths AB = 2, BC = 3, CD = 4, and DA = 5. If $\angle ABD + \angle CDB = 180°$, find the area of ABCD. Express your answer in simplest radical form.

S21

1920. Suppose at some point in time, the man is x bottle lengths away from his starting point. Then, after 5 seconds, by the Law of Cosines, the square of his distance will be

$x^2 + 4^2 - 2(x)(4)(\cos ø)$,

where ø is a random angle. Since the expected value of cos ø is 0, the expected value of the square of his distance after 5 seconds will be just $x^2 + 4^2$. Therefore, since he takes 120 steps, the answer is $120(4^2) = 1920$.

S22

$2\sqrt{14}$. Let A′ be the point such that ABDA′ is an isosceles trapezoid. Then, A′ lies on line CD since

$$\angle CDB + \angle A'DB = \angle CDB + \angle ABD = 180°.$$

Then, by Heron's Formula,

$$[ABCD] = [ABD] + [CDB] = [A'DB] + [CDB] = [A'BC] = \sqrt{7(7 - 6)(7 - 5)(7 - 3)} = \sqrt{7 * 1 * 2 * 4} = 2\sqrt{14}.$$

Q23

If $x = (64!)(63!)...(1!)$, what is the largest integer k such that $x/(2^k)$ is an integer?

Q24

Quadrilateral ABCD is cyclic with AB = 7 and CD = 8. If AC and BD intersect at E with $\angle AEB = 60°$, find the circumradius of ABCD. Express your answer in simplest radical form.

S23

1887. By Legendre's Formula, for any positive integer n,

$v_2(n!) = n - S(n)$,

where S(n) is the sum of the digits of n in binary. Therefore,

$v_2(x) = (1 - S(1)) + (2 - S(2)) + \dots + (64 - S(64)) = (1 + 2 + \dots + 64) - (S(1) + S(2) + \dots + S(64))$

$= (64 * 65)/2 - (S(1) + S(2) + \dots + S(64)) = 2080 - (S(1) + S(2) + \dots + S(64))$.

To find S(1) + S(2) + ... + S(64), we can consider each digit in binary separately. The 1s digit is 1 in 32 of these numbers, the 2s digit is 1 in 32 of these numbers, etc., until the 32s digit is 1 in 32 of these numbers, and the 64s digit is 1 in 1 of these numbers.

So, S(1) + S(2) + ... + S(64) = 6 * 32 + 1 = 193, which means the answer is 2080 - 193 = 1887.

S24

$(13\sqrt{3})/3$. Let O be the circumcenter of ABCD, and let F be a point on the circumcircle of ABCD such that BF = 8 and A and F are on opposite sides of line BO. Then, note that

$60° = \angle AEB = \angle ACB + \angle CBD = \angle ACB + \angle BCF = \angle ACF$,

which means $\angle AOF = 120°$, so $\angle ABF = 120°$. So, by the Law of Cosines, $AF^2 = 7^2 + 8^2 + 7 * 8 = 169$, so AF = 13. Since triangle AOF is isosceles with $\angle AOF = 120°$, the circumradius is $(13\sqrt{3})/3$.

Q25

Let ABC be a triangle with area 60. Also, D lies on AB such that $AD = 2DB$ and E lies on AC such that $CE = 3AE$. If lines BE and CD intersect at F, find [ADEF].

Q26

Quadrilateral ABCD is cyclic with $AB = 5$ and $CD = 9$. If AD and BC intersect at E with $\angle AEB = 30°$, find the area of the circumcircle ABCD. Express your answer in simplest radical form in terms of π.

S25

13. We will express everything in barycentric coordinates where (a, b, c) is aA + bB + cC. Firstly,
D = (1/3, 2/3, 0) and E = (3/4, 0, 1/4),
which means
F = (3/10, 3/5, 1/10) = (3/5)B + (2/5)E = (1/10)C + (9/10)D.
Therefore,
[ADEF] = [ADF] + [AEF] = [ABC](AD/AB)(DF/DC) + [ABC](AE/AC)(EF/EB)
= 60(2/3)(1/10) + 60(1/4)(3/5) = 4 + 9 = 13.

S26

106π - 45π√3. Let O be the circumcenter of ABCD, and let F be a point on the circumcircle of ABCD such that CF = 5 and D and F are on the same side of line CO. Then, note that

30° = ∠AEB = ∠CAD - ∠ACB = ∠CAD - ∠CAF = ∠DAF = ∠DCF,

which means ∠DOF = 60°, so ΔDOF is equilateral. By the Law of Cosines, DF^2 = 5^2 + 9^2 - (√3)(5)(9) = 106 - 45√3. Since ΔDOF is equilateral, we get that the area of the circumcircle of ABCD is 106π - 45π√3.

Q27

Given that u, v, w, x, y, and z are real numbers such that $u^2 + x^2 = v^2 + y^2 = w^2 + z^2 = 1$, find the minimum possible value of $uv + vw + wx + xy + yz - zu$. Express your answer as a common fraction in simplest radical form.

Q28

Find the minimum of $4a^3 + 27/(a^2)$ for all positive real numbers a. Express your answer in simplest radical form.

S27

$-(3\sqrt{3})/2$. Let $S = uv + vw + wx + xy + yz - zu$. Then, note that if $A = (u, -x)$, $B = (y, v)$, and $C = (-w, z)$, by the Shoelace Theorem,

$|S| = 2[ABC]$,

which means we must find the maximum of [ABC]. The only constraint for A, B, and C is that they must lie on the circle centered at (0, 0) with radius 1, so to maximize [ABC], we must have each point of ΔABC to be one of the arc midpoints of the other two.

Therefore, [ABC] is maximized when ΔABC is equilateral and has area $(3\sqrt{3})/4$, so the minimum of S is $-(3\sqrt{3})/2$.

S28

$5(2916)^{(1/5)}$. By AM-GM, we get that

$(2a^3 + 2a^3 + 9/(a^2) + 9/(a^2) + 9/(a^2))/5 \geq (2916)^{(1/5)}$

$(4a^3 + 27/(a^2))/5 \geq (2916)^{(1/5)}$

$4a^3 + 27/(a^2) \geq 5(2916)^{(1/5)}$.

Q29

How many ways can 10 be expressed as an ordered sum of one or more positive integers? (For example, 10, 1 + 3 + 6, and 3 + 6 + 1 are all distinct ordered sums.)

Q30

Let A, B, and C be variable points on an ellipse with a major axis of length 6 and a minor axis of length 4. Also, let M and N be the midpoints of AB and AC, respectively, and let D be a variable point not on AB, BC, CA. Moreover, E ≠ D and F ≠ D are intersections of lines DM and DN with the ellipse, respectively. If [DAB] = [EAB] and [DAC] = [FAC], find the maximum possible value of [DBC]. Express your answer as a common fraction in simplest radical form.

S29

512. We can express each ordered sum as a series of 10 Xs and some number of |s. For example,
10 would be represented as XXXXXXXXXX,
1 + 3 + 6 would be represented as X|XXX|XXXXXX,
and 3 + 6 + 1 would be represented as XXX|XXXXXX|X.
Since each pair of Xs can either have a | or not have a |, we get 2^9 = 512 ordered sums in total.

S30

(9√3)/2. Let x be a transformation that stretches the ellipse in the direction of its major axis by a factor of 1/3 and stretches the ellipse in the direction of its minor axis by a factor of 1/2. So, the transformed ellipse is a circle of radius 1. We will denote transformed points with ʻ.

Since [D′A′B′] = [E′A′B′] and [D′A′C′] = [F′A′C′], D′ must be the orthocenter of ΔA′B′C′. If we let A′D′ hit the circle at Z, then [D′B′C′] = [ZB′C′], and when [ZB′C′] is maximized, each of Z, B′, C′ is the arc midpoint of the other two.

So, [ZB′C′] is maximized at (3√3)/4 when ΔZB′C′ is equilateral, which means the maximum of [DBC] is (9√3)/2.